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**The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.**

**Sam Houston**

**Cultivated mind is the guardian genius of democracy. . . . It is the only dictator that freemen acknowledge and the only security that freemen desire.**

**Mirabeau B. Lamar**

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# THE TEXAS MATHEMATICS TEACHERS' BULLETIN

Volume XIV, Number 2

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This bulletin is open to the teachers of mathematics in Texas for the expression of their views. The editors assume no responsibility for statements of facts or opinions in articles not written by them.

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# THE TEACHING OF FRACTIONS

P. M. BATCHELDER
*The University of Texas*

The teaching of fractions in algebra seems to have received less attention in the reform movement of the present century than most other parts of the subject; it is not discussed at all, for example, in Schultze's *The Teaching of Mathematics in Secondary Schools* or J. W. A. Young's *The Teaching of Mathematics in the Elementary and the Secondary School.* There is need for reform here, however, as anyone who is familiar with the mathematical attainments of high-school graduates can testify. On a test covering twelve topics in high-school algebra and arithmetic given to freshmen in Whitman College in 1925 (reported in *The Mathematics Teacher,* vol. 19, p. 206), the grades on the part which dealt with fractions were lower than those on any of the other parts. The history of mathematics likewise shows that the subject is an intrinsically difficult one. We have considerable information from the famous Ahmes papyrus and other surviving documents regarding the state of Egyptian mathematics about 2000 B. C., and we find that the methods of handling fractions at that time were exceedingly cumbersome. It is only by a long and gradual process, extending over many centuries, that the much more efficient modern methods have been evolved.

In an attempt to get some light on the weaknesses of our present methods of teaching fractions and suggestions for improvement, the writer last year assigned the subject to one of his graduate students, Miss L. Pearl Hicks of Stephenville, Texas, who is an experienced teacher of high-school algebra. Miss Hicks gave a uniform test on arithmetical fractions to 170 pupils in the seventh grades of five schools, and a similar test on algebraic fractions to 60 pupils in the eighth grades of two schools, mostly in Dallas County. The results of these tests and Miss Hicks' conclusions from them are given in detail in her M.A. thesis, a copy of which is deposited in the library of the University of Texas.

In the seventh-grade test, the students did most poorly on the part dealing with division of fractions, the chief error being failure to invert the divisor or inverting the dividend instead. Next came addition, where the commonest error was adding all the numerators and all the denominators to get the numerator and denominator of the sum. Then came multiplication, subtraction, reducing mixed numbers to fractions, etc., in this order. In the eighth-grade test the part on addition was done most poorly, then subtraction, multiplication and division, changing mixed expressions to fractions, etc., the errors being of the same types as on the other test.

The difficulties which the students have in handling fractions and the errors which they make seem to indicate that they have been taught in too mechanical a fashion. They have memorized a set of rules for each operation, but have failed to gain any conception of the *reasons* for the rules; consequently after a brief interval they either forget the rules completely, or forget which rule applies to which operation, and having never perceived the reasonableness of the rules they fail to see the absurdity of such statements as

$$\frac{1}{2} + \frac{2}{5} = \frac{3}{7}$$

Another point brought out by Miss Hicks' investigation is that the seventh-grade pupils handled fractions more accurately on the average than the eighth-grade pupils. This indicates a failure on the part of the eighth-grade teachers to correlate their teaching of algebra properly with the knowledge of arithmetic which their students already possess. Since the operations with algebraic fractions are precisely the same as those with numerical fractions, they should not be taught as something new, with a fresh set of rules to be memorized; instead, the fact should be emphasized that the rules of operation are exactly the same, whether the numerator and denominator are known numbers or whether they are unknown numbers represented by literal expressions.

In conclusion I will quote a passage from Miss Hicks' thesis (p. 45), the significance of which extends far beyond the teaching of fractions:

> The school making the highest score in this study has retained its seventh-grade arithmetic teacher for several years. This teacher spends part of her vacation time studying new methods and observing how the teaching is done elsewhere. On the other hand, the school that made the lowest score secures teachers with only one year of college training, no practice methods, and no experience; the salary is low and a change of teachers is made frequently.

# FACTORING

EDNA VON ROSENBERG
*Austin High School*

Factoring has always been a difficult subject to teach; but it is of such fundamental importance for future successful work in mathematics, that it must not be passed over lightly. Not only does proficiency require that a student be able to recognize a product at sight, but it also requires him to be able to break it up into its prime factors.

Most boys and girls seem to have their greatest trouble in determining the type to which their factoring problems belong; and ability to recognize the types must be developed before either speed or accuracy in factoring can be attained. Various methods have been used in trying to help students distinguish one type from another. Below is a scheme which has been found satisfactory in teaching eighth and ninth grade students. The types are given in the order in which they are presented, and students are taught to follow the outline in that order when factoring examples assigned them.

*Factoring Outlines*

I. Look for a monomial highest common factor.
II. Count the terms.
    1. If there are two terms, try
        (1) Difference of two squares.
        (2) Difference of two cubes.
        (3) Sum of two cubes.
    2. If there are three terms, try
        (1) Trinomial in the form of a perfect square.
        (2) Product-sum (coefficient of $x^2$ is 1).
        (3) Cross-product (coefficient of $x^2$ is not 1).
        (4) $x^4$ type.
    3. If there are four or six terms, try
        (1) Key type.
        (2) Grouping for a polynomial highest common factor.

NOTE.—If an expression cannot be factored by any of the above methods, try the factor theorem, or vanishing-type method, particularly for expressions beginning with $x^3$.

No attempt will be made to give the method of presentation for each type, although it is probably necessary to make a few general remarks and to explain the names of some of the types. Of course, it is impossible to fit all factoring problems to the outlines, but such an outline has been very satisfactory in helping students to recognize the majority of the types with which an eighth or ninth grade student is confronted. A few problems are taught as exceptions to the general rule.

If a student follows the outline, he factors expressions which are both the difference of two squares and the difference of two cubes by the first and preferable method, the difference of two squares, that type coming first under the binomial types. Nothing has been said about the general types $x^n \pm y^n$. These types are not taught to eighth and ninth grade students in some schools, but they may, of course, be added to the outline, if it seems desirable.

Under the trinomial types, a distinction is made between the product-sum type, $x^2 + ax + b$, and the more general trinomial type, $ax^2 + bx + c$, which is called the cross-product or trial type. In the former of these types, $b$ is the product of two numbers whose algebraic sum is $a$. For example:

$$x^2 - 8x + 15 = (x - 3)(x - 5)$$

and

$$x^2 + 2x - 15 = (x - 3)(x + 5)$$

Some boys and girls like to call the latter type, $ax^2 + bx + c$, the "hunting and fishing type." Such a name is really a good one. For example:

$$8x^2 - 22x - 21 = (2x - 7)(4x + 3)$$

They hunt for the cross-product which satisfies the given conditions. Under the $x^4$ type are placed such expressions as $x^4 + x^2y^2 + y^4$ and $4a^4 - 21a^2b^2 + 9b^4$, problems beginning with $x^4$ or with $x$ to some power which is a multiple of 4. Such binomials as $a^4 + 64b^4$ and $x^8 + 4y^8$ belong to

this type, but they are taught as exceptions to the general case.

Under the four or six term types the "key type" is listed first. A particular class of problems, as for example $a^2 + 9b^2 - c^2 - 6ab$ and $a^2 + b^2 - 2mn^2 - n^4 + 2ab - m^2$, which are reducible to the difference of two squares, is placed here. Students are taught that in this type there must be three or four squares, temporarily without respect to signs, and one or two keys respectively, depending upon whether there are actually four or six terms. Such terms as $-6ab$, $-2mn^2$ and $2ab$ in the above problems are the *keys*, since they unlock the problems by telling which terms to group together. The polynomial highest common factor type, which is among the first types given in most texts, is next to last in the outline. There is a particular advantage in placing it here. The reader has probably noticed that most authors include in the practice examples of this type, only such expressions as $ax + ay - 2x - 2y$, which when grouped are of such a nature that only a monomial factor can be removed from each group. Authors write such exercises, of course, because only the monomial highest common factor type has previously been taught, but students get an incorrect idea that every time they group terms, they must take out a monomial factor from each group. If the polynomial highest common factor type is among the last types given, students are taught to factor each group of terms by any preceding method. Then, such common expressions as

$$a^2 - 4c^2 + a - 2c \text{ and } x^3 - 8 - 6x + 3x^2$$

are easily recognized as belonging to this group, and students have little or no difficulty in factoring them. In teaching this type, teachers should insist that the common factor be placed first, and that the second factor be obtained by dividing the entire expression by this common factor. For example:

$$\begin{aligned} a^2 - 4c^2 + a - 2c &= (a^2 - 4c^2) + (a - 2c) \\ &= (a + 2c)(a - 2c) + (a - 2c) \\ &= (a - 2c)(a + 2c + 1), \text{ by division.} \end{aligned}$$

# SIGNS IN MULTIPLICATION

J. W. Calhoun
*The University of Texas*

The confusion in the mind of the student when he first collides with the common saying "minus times minus gives plus" seems to call for some simple explanation of the rules governing signs in multiplication. The writer well remembers his own perplexity on being told that $-7$ was "seven less than nothing" and that "minus 5 was five less than nothing" but that seven less than nothing, on being multiplied by five less than nothing, gave the familiar number 35, and it was "thirty-five more than nothing." There are probably few teachers of algebra today who deal in such terms as those quoted above, yet it may be that some simple remarks on this matter would not be amiss.

In arithmetic $5\times 7 = 35$. In arithmetic numbers have no signs, they are neither positive nor negative. In algebra when we bring positive and negative numbers into use it is customary to regard the familiar numbers of arithmetic as positive numbers. This being true, it is natural enough to say that $(+5)\times(+7) = +35$. We express this in words by saying that the product of two positive numbers is positive. What then shall we say is the product of $+5\times(-7)$?

In our introduction of negative numbers we have introduced the idea of representing numbers by the points on a line with 0 separating the positive from the negative numbers. We have defined adding a negative number as counting toward the left (the usual negative direction). Since multiplication is a repeated addition, $+5\times(-7)$ can be interpreted as counting five sevens toward the left. This brings us to $-35$. We say, therefore, that $+5\times(-7) = -35$. What then shall $-7\times(5)$ mean? Since $7\times 5$ means counting seven fives toward the right, $-7\times 5$ should mean counting seven fives toward the left. Then $-7\times 5 = -35 = 5(-7)$. We formulate this result by saying the product of a positive and a negative number is negative.

Finally what meaning shall be assigned to $(-5)\times(-7)$? Since $-5\times7=-7\times5=-35$, it is clear that $(-7)\times(-5)$ cannot be $-35$ for then we should have $(-7)\times(-5)=-7\times5$. Dividing each number of the equation by $-7$ would give $5=-5$, an absurd result. We are left then to try $(-7)\times(-5)=+35$. Then $7\times5=(-7)\times(-5)$. But there is no more inconsistency in that equation than in the equation $4\times12=8\times6$. Since this assumption introduces no contradiction and gives a definite meaning to $(-7)\times(-5)$, we then formulate the results as follows:

$$5\times\ \ 7=(-5)\times(-7)=\ \ 35$$
$$5\times(-7)=(-5)\times(\ \ 7)=-35.$$

This is the basis of the familiar rule—*The product of two factors having like signs is positive, having unlike signs is negative.*

The extension to products of more than two factors is easy.

# THE NUMERICAL NOTATION OF THE ROMANS

ERNESTINE F. LEON
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The Romans, practical rather than original, had the happy faculty, as do many eminently successful individuals, of adopting the useful discoveries of the people with whom they came in contact. Thus, while the scientific discoveries of the Roman period were due to Greeks, the Romans brought to perfection the best system of numerical notation in use among ancient peoples, a system which was used exclusively by western Europe until 1000 A. D., five hundred years after the fall of the Roman Empire, and continued in general use until the sixteenth century.

The Roman system of reckoning like that of most civilizations was decimal, since the human race has ten fingers. A relic of an earlier sextal system may be seen in the fact that among the early Latins six hundred was the largest number considered. The regular colloquialism for "a whole lot," "about a million" was in classical times *sescenti*, six hundred. Since the Babylonians used sixty rather than a hundred as a base in calculating large numbers, we may infer that this method of figuring came to the Romans through contact with the Etruscans, whom the ancient Romans as well as many modern scholars believe to have come from Lydia in Asia Minor, adjacent to the sphere of Babylonian influence.

By the time of the earliest extant Roman inscription containing numbers, however, the Etruscans had been so thoroughly crushed that no Roman statesman would have admitted their influence on anything Roman, although the academic historians refer to it. This inscription is from the base of a *columna rostrata* (a column with the beaks of ships attached to it) set up to support a statue of Admiral Gaius Duilius for his victory over the Carthaginians at Mylae on the northern coast of Sicily in 260 B. C. It gives the number of slaves captured and the value of the booty.

The copy now extant was an archaistic reproduction of the first century A. D. found in the forum in 1565 and now in the Palazzo dei Conservatori at Rome.

The system of Roman notation, more or less standardized by the third century before the Christian era, consists of six basic symbols, I, V, X, ↓, C, and C|Ↄ for 1, 5, 10, 50, 100, and 1000, respectively. The one is the single stroke found, now vertically, now horizontally, to represent a unit among many peoples. The other characters are adaptations of Greek letters which represented sounds that did not occur in the Latin language and so were superfluous as part of the alphabet. The Romans had adopted the Greek alphabet of Chalcidice through their neighbors the Etruscans to the north and the Greeks of southern Italy. In this alphabet certain signs had a slightly different sound value from that of the same letters of the more familiar Attic alphabet; namely: ×, Attic Chi, was χ, ↓, Attic psi, was ch, and the double letter psi was written by πs. These superfluous letters have a similar numerical value in Etruscan. Therefore it appears that the Romans borrowed their number system as well as their alphabet from them.

The character χ represented 10, half of it V, 5. The hypothesis was offered by the great historian Theodor Mommsen that following the vertical bar, the pictograph of a finger representing 1, the V denoting 5 represents the thumb opposed to the four fingers and the χ for 10 symbolizes the two hands. This view, however, is not generaly accepted. The familiar L for 50 is a development from the ↓ chi, with intermediate forms ⊥ and ⊥. The original form was in use until the second century B. C. Theta, Θ, sounded th, was used for 100. It was later assimilated to the form C, the initial of the word *centum,* 100.

The symbol φ for 1000 appears as φ, ∝, as well as the form CIↃ current in the empire. This last sign was not assimilated to M, the first letter of *mille,* a thousand, until the use of uncials in manuscripts from the third to the seventh centuries of our era. Uncial m looked enough like CIↃ to be confused with it. It would have taken more

Roman imagination to associate CIↃ with a square Roman capital M. The D for five hundred is half of the CIↃ for one thousand.

With the six basic symbols, addition is decidedly simple. Addition is after all the only operation that the majority of human beings ever use to any great extent, as the method of making change employed by most clerks testifies. Every five I's become V or ten I's become X. Five X's become L or ten X's C and so on.

To add:

| | |
|---|---|
| XV | III |
| XL | I |
| XX | II |
| LXXV | I |
| Sum of each column (LX) (XV) | (VII) |
| X cancels subtractive X of next column | |
| C L V | II |

When sums beyond one thousand CIↃ were reached, multiplication was sometimes indicated by repeating the curves at the side. Thus ((I)) equals 10,000 and (((I))), 100,000. The last symbol is repeated twenty-three times on the Duilius inscription. This records that the victor captured metal to the value of 2,300,000 times the Roman *as*, or unit of value in coinage. Likewise for larger numbers a bar above the figure indicates multiplication by a thousand as $\overline{XX}$, 20,000, and additional bars at the sides show multiplication by 100,000. Usage varies in inscriptions at different periods. Naturally only capitalists and public officials were concerned with such large sums. The state as well as these millionaires had trained slaves, generally Greeks, as bookkeepers, *actuarii*.

The economical Roman saved space in representing his decimal system, lessening the repetition of the symbols for units and tens by the use of characters for 5, 50, and 500. Space was saved also by writing numbers ending in eight and nine by the subtractive method. A number to the left of a larger number meant that the larger was less by the smaller, as in the familiar IX, 9. Only the numerals I, X,

and C are so subtracted in the more exact public inscriptions. V, L, and D are never subtracted. The reason for this is obvious, since the chief advantage of the use of the method was economy of space, and nothing was accomplished by writing the next ten minus five instead of merely adding five, as XV not XVX. In official inscriptions also I is subtracted only before V and D.

The Roman mind found no difficulty with the subtractive method. This form of reckoning was used in figuring dates, as four days before the Kalends of April, *A. D. IV Kal. Apr.* for March 29. Also the Latin for eighteen and nineteen is two and one from twenty respectively and so on with the larger numbers ending in eight and nine. All forms of subtraction appear most frequently in the private inscriptions. For example in the collection at Columbia University a small epitaph to a slave girl records her age as XIIX, that is, eighteen.

The Roman system of fractions was based on the division into twelve *unciae* (from which we derive both ounces and inches) of the bronze *as*, originally a pound of bronze, the unit of value. The unit was gradually reduced to a coin the size of an English penny, and silver and gold used for coins of larger value. Of these coins the *sestertius* (half of the third) worth two and a half *asses* is written HS. The line across the two bars indicates that it is a number. The S stands for *semis* 6/12 or 1/2. All other fractions were written by a symbol derived from the initial of their name and a number of horizontal lines each meaning 1 as S= for 8/12, 2/3, i.e., 1/2+2/12. Later the *uncia* was divided into 144 parts. Of these the *scrupulus*, 1/24 of an *uncia* with its sign —), is still used in apothecaries' weight.

In spite of their adoption of a convenient number system the Romans express no regard for mathematics as abstract reasoning. It is interesting to note that arithmetic and geometry are Greek terms but calculate (to count by pebbles) is a good word of Latin origin. With the Romans, mathematics was a matter of bookkeeping and classed with the fields of its application, engineering and architecture, as a trade beneath the dignity of the military and legal sciences, the only fit occupations for a Roman aristocrat.

# FERMAT'S LAST THEOREM*

H. S. VANDIVER
*The University of Texas*

Fermat's Last Theorem, a statement which is famous in the history of mathematics, namely, that there do not exist integers $x$, $y$, and $z$, none of which being *zero*, which satisfy

$$x^n + y^n = z^n, \tag{1}$$

$n$ being a given integer $>2$. It was first given by Fermat who wrote, about the year 1637, upon the margin of his copy of the works of Diophantus, "I have discovered a truly remarkable proof which this margin is too small to contain." He did not publish his proof, however, and no complete demonstration has as yet been discovered.

The theorem was proved by Euler for $n = 3$ and 4. To prove it in general, it is then not difficult to see that it is sufficient to demonstrate the impossibility of

$$x^l + y^l + z^l = 0 \tag{2}$$

in non-zero integers $x$, $y$, and $z$ for any odd prime $l>3$. This was proved by Legendre in 1823 for $l = 5$, and later by Lebesgue for $l = 7$. These proofs excited great interest among the mathematicians of the time and many efforts were made to extend them to other values of $l$, a number of them resulting in errors. For example, the celebrated mathematicians Lamé and Cauchy published many results concerning (2) which were based on assumptions later proved false. The discussion and study of these mistakes led to the formulation, by Kummer, of one of the most powerful and fruitful concepts which has ever been introduced into mathematics, i.e., the notion of *ideal* numbers. By means of this idea Kummer was able to prove (in 1850) that (2) is impossible in integers $x$, $y$, and $z$, none zero, for

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*Reprinted from the Encyclopaedia Britannica, 14th edition, Vol. 9, pp. 173–174.

all primes $l$ for which none of the numerators of the Bernoulli numbers *(q. v.)* $B_a$, $a = 1, 2, \ldots, \frac{(l-3)}{2}$, is divisible by $l$, where $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, etc. Primes $l$ of this character are called *regular*. By extensive numerical computations he found that the only primes less than 100 which are not regular are 37, 59, and 67, and in a later article published in 1874 he obtained results from which it may be inferred that the only primes which are not regular between the limits 100 and 166 are 101, 103, 131, 149, and 157. In the year 1857 Kummer published another memoir in which he concluded after a long and complicated procedure that (2) is impossible under three assumptions concerning the nature of the algebraic field defined by $\zeta$ where $\zeta = e^{2i\pi/l}$. He finds, again by the use of extensive numerical computations, that these three assumptions are satisfied for $l = 37$, 59, and 67, and hence that (2) is impossible for all primes $l<100$.

In papers published recently by H. S. Vandiver several errors were pointed out in the arguments employed by Kummer in his paper of 1857 and the necessary corrections were made. The latter's extensive numerical computations concerning the regular and non-regular primes $l$ which are less than 100 and which we have already referred to, have not all been checked, however, by other writers. If we assume they are all correct then (2) has been proved impossible for all primes $l<100$. For the statement of later results we shall divide the discussion into two cases. If in (2), $x$, $y$, and $z$ are prime to each other and to $l$, this condition will be referred to as Case I of Fermat's last theorem; if $x$, $y$, and $z$ are prime to each other and one of them is divisible by $l$, the condition will be called Case II of the theorem. We shall also confine ourselves to mentioning only those criteria for the solution of (2) which do not involve in their enunciation the theory of algebraic numbers.

Legendre published in 1823 the following theorem due to Sophie Germain: If there exists an odd prime $p$ such that the congruence

$$u^l + v^l + w^l \equiv 0 \pmod{p} \tag{3}$$

has no set of integral solutions $u$, $v$, and $w$, each not divisible by $p$, and such that $l$ is not the residue of the $l$th power of any integer modulo $p$, then (2) has no solutions $x$, $y$, and $z$, each prime to $l$. By means of this result Sophie Germain proved that (2) is impossible in Case I for all primes $l<100$.

In 1857 Kummer proved that if (2) is satisfied in Case I then

$$B_n \left[ \frac{d^{l-2n} \log (x + e^v y)}{dv^{l-2n}} \right]_{v=0} \equiv 0 \pmod{l}, \tag{4}$$

$n = 1, 2, \ldots, \dfrac{(l-3)}{2}$, where $B_n$ is, as before, the $n$th Bernoulli number and the other symbol designates the result obtained by taking the $(l-2n)$th derivative with respect to $v$ of the logarithm of $x + e^v y$, where $e$ is the Napierian base, and setting $v = 0$. In 1905, Mirimanoff showed that the criteria (4), with the others obtained from them by replacing $x$ by $z$, etc., are equivalent to

$$\frac{B_{l-i}}{2} f_i(t) \equiv 0 \pmod{l} \tag{4a}$$

$$i = 3, 5, \ldots, l-2; \quad -t = \frac{x}{y}, \frac{y}{x}, \frac{x}{z}, \frac{z}{x}, \frac{y}{z}, \frac{z}{y};$$

$$f_i(t) = \sum_{k=1}^{l-1} k^{i-1} t^k, \tag{4b}$$

and added the criterion

$$f_{l-1}(t) \equiv 0 \pmod{l}$$

for the solution of (2) in Case I. The criteria (4a) and (4b) are now generally referred to as the Kummer criteria.

In 1908 Dickson, using Sophie Germain's theorem, proved that (2) is impossible in Case I for all primes $l<7{,}000$.

Wieferich, in 1909, by the use of (4a) and (4b) proved that, if (2) is possible in Case I, then $2^{l-1} \equiv 1 \pmod{l^2}$, and in 1910 Mirimanoff obtained, on the same assumption,

$$3^{l-1} \equiv 1 \pmod{l^2},$$

Furtwängler in 1912 proved that, if (2) is satisfied in Case I and $r$ is a factor of $x$ or of $x^2 - y^2$, where

$$x^2 - y^2 \equiv 0 \pmod{l},$$

then $r^{l-1} \equiv 1 \pmod{l^2}$. This result includes those of Wieferich and Mirimanoff just referred to. In the year 1925 Beeger found that the only primes $l<14{,}000$, for which $2^{l-1} \equiv 1 \pmod{l^2}$ are $l = 1{,}093$ and $l = 3{,}511$, the case $l = 1{,}093$ having been previously noted by Meissner. Using these results with those of Dickson he concludes that (2) is impossible in integers $x$, $y$, and $z$ prime to $l$, for all primes $l<14{,}000$. H. S. Vandiver obtained in 1926 the theorem that, if there exists an odd prime $p = 1 + ml$, such that $m<10l$ and

$$u^l + v^l + w^l \equiv 0 \pmod{p}$$

has no set of integral solutions $u$, $v$, $w$, each not divisible by $p$, then (2) has no solution in Case I. He also proved (1925) that if (2) is satisfied in Case I then

$$\ldots f_i(t) f_{l-i}(l^{1-t}) \equiv 0 \pmod{l}$$

$i = 1, 2, \ldots, l-1$; $t$ having the same meaning as in (4a).

As to Case II of the theorem, i.e., when one of the integers $x$, $y$, and $z$ in (2) is divisible by $l$, no result has been published which is known to represent an advance over Kummer's results in his memoir of 1857. In general, as to the present attitude toward Fermat's last theorem, there is evident among mathematicians a growing opinion that it is not true. In particular, several specialists in the theory of numbers think that it is quite possible that there exist odd prime integers $l$ and integers $x$, $y$, and $z$ prime to each other, $z \equiv 0 \pmod{l}$ such that

$$x^l + y^l + z^l = 0.$$

However, in Case I, everything indicates that it is true, although it has not been proved. That it has not been proved for this case is one of the most amazing facts in present-day mathematics, and for this reason alone the great celebrity of the theorem is perhaps justified. Also the many attempts to solve it by competent mathematicians have led to a number of remarkable developments in number theory, including very abstract conceptions and profound results, arrived at only after long chains of reasoning. In 1907 the Wolfskehl prize of 100,000 marks, for the first demonstration of the theorem was established. This led to the publication of several thousand erroneous "proofs" by individuals who were, for the most part, not at all equipped mathematically to cope with the problem.

BIBLIOGRAPHY.—P. Bachman, *Das Fermatproblem in seiner bisherigen Entwicklung* (1916); L. E. Dickson, *History of the Theory of Numbers,* vol. ii (1920); L. J. Mordell, *Three Lectures on Fermat's Last Theorem* (1921); E. Landau, *Vorlesungen über Zahlentheorie,* vol. iii (Leipzig, 1926); "Report on the Theory of Algebraic Numbers," II, ch. 2, *Bulletin of the National Research Council* (1928). (H. S. V.)

# THE MATHEMATICS REMEDIAL PROGRAM USED IN THE PORT ARTHUR HIGH SCHOOL

Miss Edith Benz, *Instructor*

F. D. Austin, *Director of Mathematics Department*
*Port Arthur High School*

For some time a need has been felt for short tests which would show us how successful our teaching is on sub-units of work; such as, the multiplication of a positive by a negative number, the division of a positive by a negative number, the solution of a verbal problem, the drawing of a geometric figure, and stating the "given" and "to prove." The tests have served several purposes. They have improved teaching by helping the instructor to locate the student's difficulties, by eliminating time spent on units which are mastered very quickly and thus giving more time to those which are not so easily comprehended, and by serving as a measure of student's achievement. Each component part of a unit is tested. Not only the teacher but also the student needs to be aware of his weaknesses as he progresses so that the proper remedial work can be applied. The student must have fixed clearly in mind the fundamental principles and widely used tools of mathematics in order to improve his technique.

A frequency of error sheet made from the Douglass Standard Survey Test, given to those students who have studied algebra 4½ months, served as a background for determining the units to be tested in Algebra. In Geometry, the units were determined by an analysis of errors made on standard tests. The supervised study period aided in locating the difficulties, for here the teacher has the opportunity of seeing the student's attitude toward his work and is enabled to detect his errors. Further difficulties were found through current literature, opinions of modern psychologists and educators, and through conference with the student.

The units tested in Algebra are:

1. Vocabulary or concepts used in beginning Algebra.
2. Addition and Subtraction of Signed numbers.
3. Multiplication and Division of Signed numbers.
4. Addition and Subtraction of Monomials.
5. Addition and Subtraction of Polynomials.
6. Multiplication and Division of Monomials.
7. Multiplication and Division of Polynomials.
8. Linear Equation.
9. Verbal Problem and Isolated Concepts.
10. Ability to read a bar, line, and circle graph.
11. Special products and factoring.
12. Fractions.
13. Formula.
14. Substitution in equation, formula, etc.
15. Radicals.
16. Exponents.
17. Test on like terms, numerical coefficients, etc.
18. Binomial theorem.
19. Ability to read a graph.
20. Quadratic equation.
21. Ratio and proportion.
22. Progression.
23. Verbal problems.

The units tested in Geometry are:

1. Definitions.
2. Drawing figures.
3. Stating "given" and "to prove."
4. Converses.
5. Loci.
6. Additional facts.
7. Drawing conclusions from data.
8. Judging the correction of conclusions.
9. Analyzing constructions.
10. Ability to organize facts.

Below is included a Geometry test on "Given and to Prove" and an Algebra test on "Like Terms, Numerical Coefficients, Substitution, etc."

Name ..... Teacher .....

Date ..... Score .....

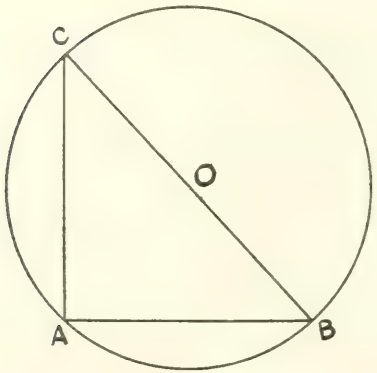
HIGH PLANE GEOMETRY

*Diagnostic Test:* "Given" and "To Prove."

*Time:*

*Directions:* State what is "given" and what is "to prove" in each of the following propositions in terms of the figure (use symbols).

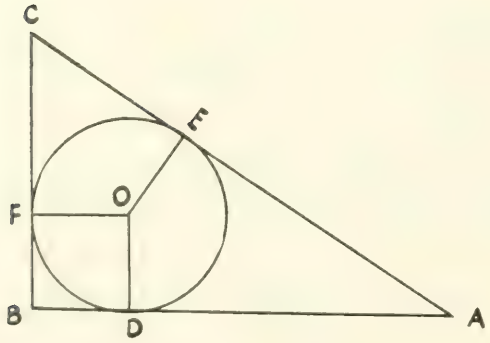
I. The center of a circle circumscribed about a right triangle, is the midpoint of the hypotenuse.




Given:

To Prove:

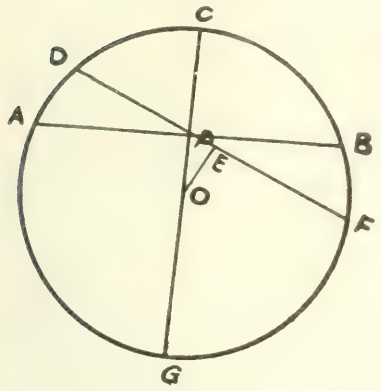
II. The diameter of a circle inscribed in a right triangle is equal to the difference between the hypotenuse and the sum of the other two sides.




Given:

To Prove:

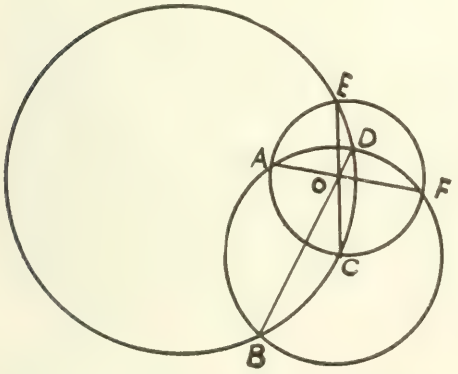
III. The shortest chord that can be drawn through a given point within a circle is that which is perpendicular to the diameter through the point.




Given:

To Prove:

IV. If three circles intersect one another, the common chords are concurrent.




Given:

To Prove:

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Name ..... Teacher .....

Date ..... Score .....

FIRST-YEAR ALGEBRA TEST

Like Terms, Numerical Coefficients, Substitution

$$3x^2 - 4xy - 3x^3y^2 + 2z + 3xy$$

1. The word which describes the above algebraic expression is _____.
2. The above expression has _____ terms.
3. It has _____ numerical coefficients.
4. It has _____ like terms.
5. If $x$ represents 4, the value of the highest power of $x$ is _____.
6. There are _____ exponents of literal factors.
7. In the third term there is implied _____ factors to be multiplied.
8. If $x$ represents 0, $y$ represents 1, and $z$ represents 2, the value of the expression is _____.
9. If $x$ represents 2, $y$ represents 3, and $z$ represents 4, the value of the expression is _____.
10. The sum of the like terms is _____.

The diagnostic tests, as mentioned above, are only the starting point for the remedial program, in that the weaknesses are located. The largest problem is that of correcting the errors and applying the proper remedial material. The frequency of error sheet is to be made after each test is given. This sheet will serve as a guide for reteaching in that the errors are analyzed and frequency of each tabulated, and as a guide for building a course of study when a number of these sheets have been filed to give a cross-section of existing conditions. The remedial material to be administered depends upon conditions found among students; it may mean reteaching, individual instruction, or drill work.

This is the first year for a program of this type to be used in the Port Arthur High School, but the median for Algebra has been raised from 7 to 14, and the Geometry has been brought to standard according to the Schorling-Sanford Achievement Tests in Geometry. Such a program can only be justified by the results obtained which we believe have proved its worth, and such a program is economical enough to be used in any high school.

# SUBJECT MATTER IN JUNIOR HIGH SCHOOL MATHEMATICS

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## I. Historical Sketch

Dr. E. C. Broome, who made a study of the three-year curriculum used in Harvard College about 1650, reports that "the college course included no mathematics until the senior year, and then only the elements of arithmetic and geometry."[1] It required almost one hundred years for arithmetic to pass from a senior study to the level of an admission requirement. Continuing the study, Broome states: "Since arithmetic appeared as an entrance requirement at Yale for the first time in 1745, it is probable that the new subject had not become sufficiently well-established by 1748 to have a place among the requirements for admission to Princeton."[2]

According to Broome, "The laws of Harvard College for 1820 make another important addition to the list of subjects required for admission: 'Algebra to the end of simple equations, comprehending also the doctrines of roots and powers, arithmetical and geometrical progression.' . . . Columbia followed with algebra in 1821, Yale in 1847, and Princeton in 1848. . . . Geometry was also made an entrance subject at Yale in 1856, at Princeton, Michigan, and Cornell in 1868, and at Columbia in 1870."[3]

In 1888, President Eliot of Harvard University made an address before the National Education Association in which he claimed that high school, college, and professional training should be completed so that young men might enter practical life at an earlier age. In 1892, the Committee of Ten was appointed with Dr. Eliot as chairman.

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[1]E. C. Broome, *A Historical and Critical Discussion of College Admission Requirements*, p. 22.

[2]*Ibid.*, pp. 31–32.

[3]*Ibid.*, pp. 43, 45.

One year later, the Committee made a report in which it recommended that certain college-preparatory subjects, including algebra, be begun in the upper grades of the elementary schools. In 1895, the Committee of Fifteen, appointed to study the elementary program, accepted the proposal of the Committee of Ten with regard to moving elementary algebra to a lower level of the school system. As already stated, arithmetic had passed from the senior year in college to become an entrance requirement. Later, it became a subject for study in every one of the elementary school grades.

But with arithmetic and algebra moved into the grades which were later to become included in the junior high school, it was necessary to await the coming of the new segment of the school system before geometry became a subject for study below the senior high school level.

## II. Textbooks and Content

In the last half of the nineteenth century, the textbooks in elementary mathematics became burdened with subject matter that had been developed to meet the needs of earlier times, but for which there was no longer a legitimate use. This was especially true with the textbooks in arithmetic. It was not an uncommon thing for pupils to be required to study arithmetic for seven or eight years, or more, and to complete three or four textbooks of generous size. As an example of the more or less common practice, the writer may cite his own experience with arithmetics. The four books used were Ray's *Primary Arithmetic,* Ray's *Intellectual Arithmetic,* Ray's *Third Part,* and Ray's *Higher Arithmetic.* It so happened that Ray's *Higher Arithmetic* was revised before the writer escaped from the elementary school, and so he was confronted with the necessity of "working through" two editions of the advanced book. Similar conditions prevailed with regard to textbooks in algebra and geometry. In making preparation for college, two books in algebra were often used; for example, Well's *Academic Algebra* and Well's *University Algebra.* Plane

and solid geometry textbooks often contained, in addition to a liberal supply of basic theorems, as many as seven or eight hundred original exercises. (For example, see Wentworth's *Plane and Solid Geometry*, 1893.) But it was not only the quantity but the quality of subject matter that aroused a militant call for change.

In 1899, the Committee on College Entrance Requirements made the following statement: "We recommend that the course in arithmetic required of all students be limited, roughly speaking, to the following topics: the four fundamental operations for integers, and common and decimal fractions; the most important weights and measures; percentage and its application to simple interest; and that it be completed in the sixth grade." (Committee Report, p. 21.) For the arrangement of the remainder of the work in mathematics, the following suggestions were made by the Committee: Seventh grade—concrete geometry and algebra; eighth grade—introduction of demonstration geometry and algebra; ninth and tenth grades—algebra and plane geometry; eleventh grade—solid geometry and plane trigonometry; twelfth grade—advanced algebra and reviews.

## III. The University of Chicago Movement

In 1903, following in line with the recommendations of the Committee on College Entrance Requirements, a fusion course in mathematics was undertaken in the high school of the University of Chicago. Since the junior high school, as such, had not come into existence at that time, the work proposed was planned to meet the needs of the upper high school grades. Materials for the first level were tried out for three years before they were published, in book form, in 1906. The second book of the series did not appear until 1910; and the third book was added in 1917. Experimentation, however, was going on all of the time.

In order to give some idea of the nature of the materials, chapter headings in Book One may be cited.

Book One: The straight line, addition and subtraction; the equation; angles; areas and volumes; angle-pairs;

parallel lines; measurement of lines in space; similar figures; ratio; variation, proportion; congruent triangles; construction, symmetry, circles; positive and negative numbers; the laws of signs; addition and subtraction (algebra); multiplication and division; special products, factoring, quadratic equations; problems in one unknown; linear equations; formulas; review.

The mingling of algebra and geometry, not only in the same book but to some extent by alternation of chapters, may be readily seen. Contrary to the usual practice, concrete geometry, in the main, precedes the work in algebra.

While Book Two deals more especially with the assumptions, theorems, constructions and proofs in geometry, some attention is given to trigonometric ratios, radicals, quadratics, and fractional equations. The chapter headings in Book Three reveal a blending of algebra, trigonometry, and solid geometry.

Topics in Book Three: Functions, equations in one unknown; trigonometric functions; linear equations; quadratic equations in one unknown; factoring; exponents, radicals, irrational equations; logarithms, slide rule; logarithms of trigonometric functions; binomial theorem; equations in two unknowns in quadratics; areas of surfaces; volumes; polyhedral angles; tetrahedral angles; spherical polygons; review of work in first two years.

## IV. Recommendations of the National Committee on Mathematical Requirements

By 1921, the junior high school had made considerable headway, and many mathematics teachers had become advocates of the fusion course for the junior high school grades. Through Bulletin No. 32, 1921, published by the National Bureau of Education, the National Committee on Mathematical Requirements suggested five alternative plans for the distribution of the different phases of mathematical instruction, as follows:

PLAN A

First year: Applied arithmetic, intuitive geometry.
Second year: Algebra, applied arithmetic.
Third year: Algebra, trigonometry, demonstrative geometry.

PLAN B

First year: As in Plan A.
Second year: Algebra, intuitive geometry, trigonometry.
Third year: Applied arithmetic, algebra, trigonometry, demonstrative geometry.

PLAN C

First year: Applied arithmetic, intuitive geometry, algebra.
Second year: Algebra, intuitive geometry.
Third year: Trigonometry, demonstrative geometry, applied arithmetic.
(Algebra confined to first two years.)

PLAN D

First year: Applied arithmetic, intuitive geometry.
Second year: Intuitive geometry, algebra.
Third year: Algebra, trigonometry, applied arithmetic.
(Demonstrative geometry omitted.)

PLAN E

First year: Intuitive geometry, simple formulas, elementary statistics, arithmetic.
Second year: Intuitive geometry, algebra, arithmetic.
Third year: Geometry, numerical trigonometry, arithmetic.

## V. The Denver Outline for Junior High School Mathematics

Since the publication of the recommendations of the National Committee on Mathematics Requirements, many cities have outlined work for the different grades. Courses from the following cities have been examined: Baltimore, Denver, Detroit, Fresno, Lexington, Nashville, Oakland, Philadelphia, Richmond, St. Louis, and Trenton. The Denver outline is as follows:

Low seventh grade:
1. Fundamental processes and drill.
2. Percentage.
3. Linear measurements.
4. The graph.
5. Angles.
6. Geometric construction and design.

High seventh grade:
1. Perimeters and areas.
2. Surfaces and volumes of solids.
3. Simple accounts and business forms.
4. Interest.
5. Banking.

Low eighth grade:
1. Review of mechanics of arithmetic.
2. Algebraic language.
3. The equation.
4. Similar triangles.
5. The use of the right triangle.

High eighth grade:
1. Review of percentage.
2. Insurance.
3. Taxes.
4. Investments and thrift.
5. Positive and negative numbers.

Low ninth grade:
1. Numbers that change together.
2. The equation.
3. The right triangle and trigonometric ratios.
4. Positive and negative numbers.
5. More difficult equations.
6. Special products and factoring.

High ninth grade:
1. Algebraic fractions.
2. Literal equations or formulas.
3. Simultaneous linear equations.
4. Variation and proportion.
5. Roots and powers.
6. Powers and exponents.
7. Quadratic equations.
8. Statistics.
9. Optional topics.

This outline presents the following phases of mathematics in the several grades: Low seventh, arithmetic and concrete geometry; high seventh, geometry and arithmetic; low eighth, arithmetic, algebra, and geometry; high eighth, arithmetic; low ninth, algebra and trigonometry; high ninth, algebra and statistics.

## VI. Contents of Recent Junior High School Textbooks

The series of textbooks resulting from the fusion courses developed at the University of Chicago has been discussed. As already stated, these books were intended for the use of pupils somewhat above the junior high school level. However, in 1917, Wentworth, Smith, and Brown presented a series of three books prepared especially for the junior high school grades. Following this series in rapid succession, other series of junior high school texts have appeared. As representative of the content of the early books, the Stone series[4] may be taken.

Book I (1919): Review of methods in computation; graphs; personal, home, and community interests; industries and interest; thrift and economy; business terms, forms, and problems; investments; geometrical constructions; measuring areas; formulas; drills in computation. (Largely arithmetic and concrete geometry but with elements of algebra.)

Book II (1920): Review of arithmetic, short cuts; the formula; the equation; ratio and proportion; similar figures; trigonometric ratios; graphic methods; measurements, constructions, and observations; square root and the Pythagorean theorem; general discussion of percentage; business forms, terms, and problems; banking; methods of investing money; insurance, taxes; reviews. (Includes elements of arithmetic, geometry, algebra, and trigonometry.)

Book III (1921): Letters used to represent numbers; how to solve and use the equation; lines and angles; properties and use of the triangle; positive and negative numbers; multiplication and division of signed numbers; algebraic factors and fractions; indirect measurements; equa-

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[4]Stone, John C., *Junior High School Mathematics.* Books I, II, III. Benj. H. Sanborn & Co.

tions with two unknowns; graphs; roots and radicals; quadratic equations; properties and variation; relations expressed by fractions and per cent; interpreting statistics. (Interpretations and applications of different phases of elementary mathematics.)

Mr. C. P. Strickland[5] found the numbers of pages given to the different phases of junior high school mathematics to vary with different authors, as shown in the following table:

CONTENTS OF CERTAIN TEXTBOOKS ON JUNIOR HIGH SCHOOL MATHEMATICS

| Authors | Total Pages | Arithmetic | Algebra | Geometry | Trigonometry |
|---|---|---|---|---|---|
| Wenthworth, Smith, and Brown | 782 | 241 | 218 | 270 | 22 |
| Stone | 669 | 292 | 255 | 89 | 5 |
| Lindquist | 715 | 326 | 270 | 92 | 0 |
| Gugle | 704 | 222 | 160 | 261 | 35 |
| Schorling and Clark | 878 | 143 | 363 | 236 | 0 |

It may be noted that the total pages exceed in number the sum of the pages given to the several phases. This discrepancy is due to the fact that some pages of each book could not be classified under the headings used.

Considerable variation as to the importance of arithmetic, geometry, etc., appears in the table. Lindquist gives 326 pages to arithmetic and only 92 pages to geometry, while Schorling and Clark give 236 pages to geometry and only 143 pages to arithmetic. It is clear that textbooks in junior high school mathematics have not been standardized with respect to the relative importance of the different phases of work.

A further study of the space given to such general topics as fractions, percentage, short cuts, positives and negatives,

[5]C. P. Strickland, *Materials and Methods in Elementary Mathematics*. Unpublished Master's Thesis on file in the Library of the University of Texas.

equations, factoring, graphs, triangles, circles, volumes, etc., shows wide variation among authors. At present, it would seem that we are in the midst of a period of experimentation with fusion courses, from which experimentation it does not seem too much to hope that we shall arouse an enlivened and more vital interest in the study of elementary mathematics.

## VII. Series of Textbooks in Junior High School Mathematics


1. Bonser-Pickell-Smith, *Practical Mathematics for Junior High Schools.* Books I, II, III. Mentzer, Bush & Co., Chicago, 1924.
2. Drushel and Withers, *Junior High School Mathematics Essentials.* Books I, II, III. Lyons and Carnahan, Chicago, 1924.
3. Breslich, E. R., *Junior Mathematics.* Books I, II, III. The Macmillan Co., New York, 1925.
4. Gugle, Marie, *Modern Junior Mathematics.* Books I, II, III. Gregg Publishing Co., Chicago, 1920.
5. Hart, W. W., *Junior High School Mathematics.* Books I, II, III. D. C. Heath & Co., Boston, 1921–1925.
6. Lindquist, Theodore, *Junior High School Mathematics.* Books I, II, III. Charles Scribner's Sons, New York, 1920.
7. Schorling-Clark, *Modern Mathematics.* World Book Co., 1924.
8. Stone, John C., *Junior High School Mathematics.* Books I, II, III. Benj. H. Sanborn & Co., Chicago, 1919.
9. Taylor and Allen, *Junior High School Mathematics.* Books I, II, III. Henry Holt & Co., New York, 1923.
10. Thorndike, E. L., *Junior High School Mathematics.* Books I, II, III. Rand, McNally & Co., Chicago, 1925.
11. Vosburgh and Gentleman, *Junior High School Mathematics.* Books I, II, III. The Macmillan Co., 1918.
12. Wentworth, Smith, and Brown, *Junior High School Mathematics.* Books I, II, III. Ginn & Co., 1917.

# THE FORMULAS BASIC IN EDUCATIONAL MEASUREMENTS

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*Austin High School*

Teachers are interested in measuring the results of their efforts, and student accomplishment is taken as the basis for measurement. Tests and test scores play an important part in educational procedure. The modern teacher has the criteria to determine the degree of accuracy of a test as a measure of student accomplishment. He needs to make use of the standardized educational and the new type objective tests. He must also make use of the contributions of the mathematician, who is the instrument-maker of the statistical industry. The formulas which the mathematician has devised are for the investigator, the educator, the student of research, and the teacher. A teacher can become a skilled worker by learning to master the formulas which the mathematician has produced. He must learn how to use them, when to use them, how to tabulate data, and how to secure proper interpretation of results. It is the purpose of this discussion to set forth the formulas basic in the evaluation of the results of teaching and in determining the degree to which teaching is effective.

## I. The Measures of Central Tendency

1. *Median.*—The median, for ungrouped data, when the frequency is odd or even, may be defined as the middle of a set of scores or the average of the two middlemost scores. A survey of textbooks on measurements in education discloses different methods for determining the median for grouped data. A formula that seems to be the favorite is borrowed from Lincoln,[1] and the solution with a check will be made from Table I.

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[1]Lincoln, E. A., *Beginnings in Educational Measurements.* Lippincott Company, Philadelphia, Pa. 1924. Page 134.

TABLE I

SCORES ON SCHORLING-SANFORD ACHIEVEMENT TEST IN PLANE GEOMETRY, FORM B

| Score | Tabulation | $f$* | P. S. |
|---|---|---|---|
| 52–56 | // | 2 | 2 |
| 52 | /// | 3 | 5 |
| 48 | //// / | 6 | 11 |
| 40–44 | //// /// | 8 | 19 |
| 40 | //// //// | 9 | 34† |
| 36 | //// /// | 8 | 25 |
| 28–32 | //// / | 6 | 17 |
| 28 | //// | 5 | 11 |
| 24 | //// | 4 | 6 |
| 16–20 | // | 2 | 2 |

*$f$ = frequency; P. S. = partial sum.
†The P. S. of 34 contains the median.

Formula I. Median $= l.l. + \left(\dfrac{N/2 - \text{P. S.}_{up}}{f_{mdn}}\right) s$, where $l.l.$ is the lower limit of the interval containing the median; N/2 is half the frequency; P. S.$_{up}$ is the partial sum up to the interval containing the median; $f_{mdn}$ is the frequency of the median; $s$ is the length of the step; Mdn. is the median.

$$\text{Mdn.} = 36 + \left(\frac{26.5 - 25}{9}\right) 4 = 36.67$$

The formula is mathematically correct, is easily executed, can be applied to determine quartiles and percentiles, and computation may be checked quickly.

The result of the formula may be checked as follows:

Mdn. $= u.l. - \left(\dfrac{N/2 - \text{P.S.}_{do}}{f_{mdn}}\right) s$, where $u.l.$ is the upper limit of the interval containing the median; P.S.$_{do}$ is the partial sum down to the median.

$$\text{Mdn.} = 40 - \left(\frac{26.5 - 19}{9}\right) 4 = 36.67$$

The median should be used when quick computation is important, when extreme and erroneous scores are not to influence the results, when there is to be a comparison of standards, and when norms are to be established.

*a. Quartiles and Percentiles.*—The quartiles and percentiles are determined by Formula I. To compute the third quartile, $Q_3$, use the formula for determining the median but instead of N/2 substitute 3N/4, and for the first quartile, $Q_1$, N/4. Likewise, to compute the percentiles, if 90 per cent is wanted, use 9N/10 and for 10 per cent, use N/10.

From the data of Table I, $Q_3$ is 42.88; $Q_1$ is 29.33; 90 per cent is 47.8; and 10 per cent is 23.3.

2. *Mean* (average).—The mean is the sum of the values of the scores divided by the number of scores, or $M = \frac{\Sigma x}{N}$, where M is the mean; $\Sigma x$ is the sum of all the scores; and N is the number of scores.

Since there are several methods of determining the mean, the formula suggested is the one that Garrett terms the "short method."[2] It is also known as the + and — non-step deviation method.

The data as taken from Table I appear thus:

| Score | *f* | *d* | *fd* |
|---|---|---|---|
| 52–56 | 2 | 4 | 8 |
| 52 | 3 | 3 | 9 |
| 48 | 6 | 2 | 12 |
| 40–44 | 8 | 1 | 8 |
| 40 | 9 | 0 | 0 |
| 36 | 8 | — 1 | — 8 |
| 32 | 6 | — 2 | — 12 |
| 24–28 | 5 | — 3 | — 15 |
| 24 | 4 | — 4 | — 16 |
| 16–20 | 2 | — 5 | — 10 |
| Total | 53 | | — 24 |

Formula II. $M = M' + \left(\frac{fd}{N}\right)s$, where M is the mean;

[2]Garrett, H. E., *Statistics in Psychology and Education.* Longmans, Green and Company, New York. 1926. Page 37.

M′ is the mid-point of the interval containing the mean; *fd* is the frequency deviation; *s* is the length of the step; and N is half the frequency.

$$M = 38 + \left(\frac{-24}{26.5}\right)4 = 36.19$$

Formula II is a favorite, because there is a minimum of multiplication and of addition; and from the arrangement of the frequencies and the deviations, the standard deviation ($\sigma$, sigma) may be computed quite readily.

The mean should always be used when correlations and statistical investigations are made, when low unreliability is sought, when every score is to influence the average, and when accuracy is desired.

3. *Mode.*—The mode is variously defined:

(1) "The mode is the place of densest frequency in a distribution."

(2) "The mode is the score which appears the greatest number of times in the result."

(3) "The mode is the most frequent score or the mid-point of the step with the largest frequency."

For example, in the following scores: 47, 49, 51, 52, 52, 54, 55, 57, 60, 62, 63, 63, 63, 65, 65, 66, 66, 67, 68, 70, 71, 71, 72, 72, 73, 75, 77, and 79, the score of 63 is the mode; but when the scores are distributed in intervals of five points, thus:

| Score | 45–50 | 50–55 | 55–60 | 60–65 | 65–70 | 70–75 | 75–80 |
|---|---|---|---|---|---|---|---|
| Frequency | 2 | 4 | 2 | 5 | 6 | 5 | 3 |

then the mode is 67.5. By widening or narrowing the class interval, the mode may be made to shift along the scale. A definition for mode must provide for the distributed as well as the undistributed form. It is, therefore, evident that the mode "is the most frequent score or the mid-point of the step with the largest frequency."

The mode plays a minor part in statistical procedure and is used to find the quickest measure of central tendency.

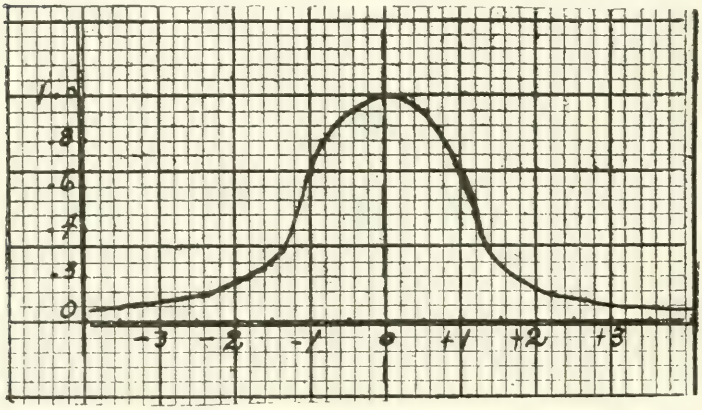
The Pearson formula for mode, for all moderately skewed distributions, is this:

Formula III. $M_o = M - 3(M - Mdn.)$, where $M_o$ is the mode; M is the mean; and Mdn. is the median.

From the data of Table I, the mode is 38; by the use of the formula, the mode is 37.63.

4. *Skewness.*—The figure gives the graph for $y = e^{-\frac{1}{2}x^2}$. The curve is bell-shaped and is called the normal probability curve. In this curve median, mean, and mode are equal. However, the scores resulting from a test given to a class

THE NORMAL PROBABILITY CURVE




$$y = e^{-\frac{1}{2}x^2}$$

when plotted, seldom approach the normal. The curve peaks either to the right or to the left of the normal and is said to skew. Pearson has given a formula for skewness as follows:

$$S = \frac{M - M_o}{\sigma};$$

but $M_o$, for moderately skewed distributions, is equal to $M - 3(M - Mdn.)$; hence,

$$S = \frac{M - M + 3(M - Mdn.)}{\sigma} \text{ and}$$

Formula IV $$S = \frac{3(M - Mdn.)}{\sigma}.$$

It will be noted that when the mean exceeds the median, the peak of the curve will be to the left of the normal; and skewness is positive. When the median exceeds the mean, the peak of the curve will be to the right of the normal; and skewness is negative. There are many causes for skewness in a distribution, but an extreme positive skewness in test scores is due to the difficulty of the test. When test scores result in positive skewness, there is excellent discrimination for the brighter and very poor discrimination for the slower pupils. Likewise, an extreme negative skewness in a distribution of test scores is indicative that the test was too easy. When test scores are of negative skewness, there is little discrimination for the better students but good discrimination for the slower students. An ideal test is one in which test scores distribute in conformity to the normal. We need to pause, however, to say that it is good psychology occasionally to give tests that are not very difficult. Every child will thereby be given an opportunity to achieve. Success gives zest to the work and stimulates the will and the ambition.

The reader is to note that variability is a factor of much importance in educational measurements and the formula for skewness is quite dependent upon a measure of variability or spread of distribution. An attempt will be made to give the formulas and to state the meaning of the measures of spread.

## II. The Measures of Variability or Spread

1. *The Standard Deviation* ($\sigma$).—The standard deviation is the measure of variability of a distribution. In an interval of $-1\sigma$ to $+1\sigma$ there will be 68.28 per cent of the scores of a distribution. This $\sigma$, standard deviation, is determined by

Formula V $$\sigma = \left( \sqrt{\frac{fd^2}{N} - c^2} \right) s,$$

where N is the total frequency; $fd^2$ is the product of $d$ and $fd$, as given in the "short method" for the mean; $c$ is $\frac{fd}{N}$; $s$ is the length of the step.

If the data of Table I are used, we have:

| Score | $f$ | $d$ | $fd$ | $fd^2$ |
|---|---|---|---|---|
| 52–56 | 2 | 4 | 8 | 32 |
| 52 | 3 | 3 | 9 | 27 |
| 48 | 6 | 2 | 12 | 24 |
| 44 | 8 | 1 | 8 | 8 |
| 36–40 | 9 | 0 | 0 | 0 |
| 36 | 8 | —1 | — 8 | 8 |
| 32 | 6 | —2 | —12 | 24 |
| 28 | 5 | —3 | —15 | 45 |
| 24 | 4 | —4 | —16 | 64 |
| 16–20 | 2 | —5 | —10 | 50 |
| Total | 53 | | —24 | 282 |

$$\sigma = \left( \sqrt{\frac{282}{53} - \left(\frac{-24}{53}\right)^2} \right) 4 = 4\sqrt{4.869} = 8.9$$

Since the mean is 36.19 and $\sigma = 8.9$, we are enabled to say that 68.26 per cent of all the scores fall within the interval 36.19 $\pm$ 8.9 or between 27.29 and 45.09.

2. *The Quartile Deviation.*—The range from $Q_1$ to $Q_3$ is called the interquartile range. $Q = \frac{Q_3 - Q_1}{2}$ and is in reality the semi-interquartile range, called, however, for brevity, the quartile deviation. The point from which Q is taken, if considered as a deviation, is: $Q_3 - \frac{Q_3 - Q_1}{2}$ and always includes 50 per cent of the measures or scores of a distribution. Since $Q_3$ and $Q_1$ have been computed and found to be 42.88 and 29.33, respectively, Q is 6.78.

(a) The probable error is to be determined from: Formula VI. P.E. $= .6745\sigma$, and can not be measured independently of the standard deviation.[3] The median $\pm$1P.E.

[3]Kelley, T. L., *Statistical Method.* Macmillan Co., New York, Page 211.

contains 50 per cent of the measures when the distribution is normal. Hence the quartile deviation and the probable error should not be considered as being the same. The probable error describes the variability or spread of the normal distribution about its median; and one is enabled to say that the chances are one to one that any item selected at random from the distribution will be included within the limits of the interval. Since the median is 36.67 and P.E. is 6, the interval is 30.67 to 42.67.

3. *The Mean Deviation.*—It is accepted that the mean deviation is equal to $0.7979\sigma$. In actual practice it is often taken as $0.8\sigma$. The mean deviation should be derived from the standard deviation. Since $\sigma$ is 8.9, the mean deviation is 6.10, and we have 57 chances in a 100 that any item selected at random will be within the limits $36.19 \pm 6.10$ or between 30.09 and 42.39.

4. *The Coefficient of Variation.*—It is often of interest to measure the relative variability of two groups on the same test or of a group on two separate tests. The comparison may be made by the use of Pearson's Coefficient of Variation Formula:

Formula VII $$V = \frac{100\sigma}{M},$$

where V is the variability; $\sigma$, standard deviation; M is the mean.

For example, if on one test, $\sigma_1$ is $x$ and on another test $\sigma_2$ is $y$, then

$$V_1 = \frac{100x}{M_1} = m \text{ and } V_2 = \frac{100y}{M_2} = n.$$

The percentage of variability that exists between the groups on a test, or within a group on two separate tests is definitely expressed by $m/n$.

## III. The Reliability of Measures

It is necessary to have a clear conception of the reliability of a measure. The underlying formulas will be presented, their application made, and the interpretation stated.

1. The standard error of the mean, $\sigma_{av.}$ is determined by:

Formula VIII $$\sigma_{av.} = \frac{\sigma_1}{\sqrt{N}},$$

where N is the total frequency; $\sigma_1$ is the standard deviation of the distribution. Since $\sigma_1$ is 8.9, $\sigma_{av.}$ is 1.22; and we may say that there are 68 chances in a 100 that the true mean is within the interval $M \pm \sigma_{av.}$, or between $36.19 \pm 1.22$. There are 99 chances in a 100 that the true mean lies within the interval $M \pm 3\sigma_{av.}$ or $36.19 \pm 3.66$.

(a) The probable error of the mean, $P.E._{av.}$, is determined by:

Formula IX $$P.E._{av.} = \frac{.6745\sigma_1}{\sqrt{N}}.$$

In our problem $P.E._{av.}$ is .83, and the interpretation is that there are 99 chances in 100 that the mean lies within the interval $M \pm 4P.E._{av.}$ or $36.19 \pm 3.32$. The difference between the intervals $M \pm 3\sigma_{av.}$ and $M \pm 4P.E._{av.}$ is due to the fact that $3\sigma_{av.}$ is equal to $4.45P.E._{av.}$. Since the reliability of measures is determined from $\sigma$, $\pm 3\sigma_{av.}$ should form the interval for the true mean.

2. The standard error of the median, $Q_{mdn.}$ is $5/4\sigma_{av.}$, and there are 50 chances in 100 that the true median is within the interval $Mdn. \pm 5/4\sigma_{av.}$.

3. The standard error of the standard deviation, $\sigma_\sigma$, is is determined by:

Formula X $$\sigma_\sigma = \frac{\sigma_1}{\sqrt{2N}};$$

and the interpretation is that there are 68 chances in 100 that $\sigma$ lies within the interval $\sigma \pm \sigma_\sigma$ or $8.9 \pm .86$, and there are 99 chances in 100 that $\sigma$ lies within $\sigma \pm 3\sigma_\sigma$.

## IV. Correlation

Correlation is a measure of the degree of parallelism between two sets of scores or obtained data. Where $N<25$, the reader is referred to the rank method of correlation as presented by Holzinger.[4] However, where N is 30 or more cases, the Pearson Product-Moment Method should be used. The scatter diagram for the product-moment method is essentially the setting of two sets of scores, by the "short method" of finding the mean, in a horizontal and in a vertical position on the same plot. The coefficient of correlation is computed by the use of Pearson's formula:

Formula XI
$$r = \frac{\frac{\Sigma xy}{N} - c_x c_y}{\sigma_x \cdot \sigma_y};$$

where $\Sigma xy$ is the sum of the deviations; $c_x$ is the correction for the mean on the horizontal scores; $c_y$ is the correction for the mean on the vertical scores; $\sigma_x$ is the standard deviation on the horizontal scores; $\sigma_y$ is the standard deviation on the vertical scores.

Since the reliability of $r$ is $(1 - r^2)/\sqrt{N}$,

we have

Formula XII
$$\text{P.E.}_r = \frac{.6745(1 - r^2)}{\sqrt{N}};$$

and the interpretation may be made that the chances are even that true $r$ lies within the interval $r \pm \text{P.E.}_r$, and there are 99 chances in 100 that true $r$ lies within $r \pm 4\text{P.E.}_r$. To be certain that there is some correlation, an obtained $r$ should be equal to or greater than $4\text{PE.}_r$.

The following scatter diagram is given as a basis for illustrating correlation and the computation of $r$ and $\text{P.E.}_r$:

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[4]Holzinger, K. J., *Statistical Methods for Students in Education.* Ginn and Company, Boston. Pages 143–154.

CORRELATION: SCHORLING-SANFORD ACHIEVEMENT TEST IN PLANE GEOMETRY, FORM B, AND I.Q., OTIS HIGHER EXAMINATION, FORMS A AND B

OTIS I.Q.

| | 93 | | 99 | | 105 | | 111 | | 117 | | 123 | | 129 | $f$ | $d$ | $fd$ | $fd^2$ | $xy$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 52. | | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | 1 | 1 | 2 | 7 | 14 | 98 | 77 |
| 49. | | | | | | | | | | 1 | 1 | 1 | | 3 | 6 | 18 | 108 | 72 |
| 46. | | | | | | | | | 1 | | 2 | | | 3 | 5 | 15 | 75 | 50 |
| 43. | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | | 9 | 4 | 36 | 144 | 68 |
| 40. | | | | | | 1 | 2 | 1 | | 2 | | | 1 | 7 | 3 | 21 | 63 | 36 |
| 37. | | | 1 | 1 | 2 | 1 | 1 | 1 | | 1 | | | | 8 | 2 | 16 | 32 | —16 |
| 34. | 1 | | | 1 | 3 | 1 | | | 1 | | | | | 7 | 1 | 7 | 7 | —14 |
| 31. | | | 1 | 1 | 1 | 2 | 5 | 1 | 2 | | | | | 13 | 0 | 00 | 00 | 00 |
| 28. | | 1 | 2 | 2 | | | | 1 | | | | | | 6 | 1 | 6 | 6 | 18 |
| 25. | | 2 | 3 | | 1 | 2 | | | | | | | | 8 | 2 | 16 | 32 | 52 |
| 22. | | | 1 | 2 | 1 | 1 | | | | | | | | 5 | 3 | 15 | 45 | 39 |
| 19. | | | 2 | 2 | 2 | | | | | | | | | 6 | 4 | 24 | 96 | 108 |
| 16. | | 2 | 1 | | | | | | | | | | | 3 | 5 | 15 | 75 | 56 |
| 13. | | | | | | | | | | | | | | 0 | 6 | 0 | 00 | |
| $f$ | 1 | 5 | 11 | 9 | 11 | 9 | 9 | 5 | 5 | 5 | 4 | 4 | 2 | 80 | | 51 | 781 | 576 |
| $d$ | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | | | | | |
| $fd$ | 6 | 25 | 44 | 27 | 22 | 9 | 0 | 5 | 10 | 15 | 16 | 20 | 12 | —59 | | | | |
| $fd^2$ | 36 | 125 | 176 | 81 | 44 | 9 | 0 | 5 | 20 | 45 | 64 | 100 | 72 | 777 | | | | |

$$c_x = \frac{+51}{80} = 0.625; \quad c_y = \frac{-59}{80} = -0.684; \quad c_x c_y = -0.428$$

$$\sigma_x = \sqrt{\frac{781}{80} - 0.625^2} = 3.06; \quad \sigma_y = \sqrt{\frac{777}{80} - 0.684^2} = 3.04$$

$$\sigma_x \sigma_y = 9.3$$

$$r = \frac{\frac{576}{80} - (-0.428^2)}{9.3} = 0.81$$

and

$$\text{P.E.}_r = \frac{.6745(1 - .81^2)}{\sqrt{80}} = 0.026.$$

Therefore

$$r = .81 \pm .026.$$

The coefficient of correlation, $r$, is an abstract number varying between $-1$ and $+1$, and is used to express the relationship which exists between two sets of facts that are paired. For a correlation, the standard error of estimate, $\sigma_{est.}$, is determined by Formula XIII,

$$\sigma_{est.} = \sigma_2\sqrt{1 - r^2}$$

and furnishes a basis for interpretation. It will be noted that as $r = 0$, the prediction has a standard error equal to the standard deviation of the variable; but as $r$ increases, the predictive value increases and at $r = .866$, $\sqrt{1 - r^2}$ is equal to .50 and the prediction is 50 per cent better than random guess. The probable error of estimate, the measure of departure from perfect correlation, is

Formula XIV

$$\text{P.E.}_{est.} = .6745\sigma_2\sqrt{1 - r^2}$$

where $\sigma_2$ is the standard deviation obtained from the distribution of the second test, and $\sigma_2\sqrt{1 - r^2}$ predicts the second score from a perfect correlation with the first as a true score.

(a) *The Index of Reliability.*—The correlation between a single test and the true score in the trait measured by a single test is equal to

Formula XV

$$r_{1t} = \sqrt{r_{1,2}},$$

and this $\sqrt{r_{1,2}}$ is called the Index of Reliability. It is the maximum correlation which can ever be obtained with the

test. Since $r_{1,2}$ in the correlation given was .81, and the index of reliability, $\sqrt{.81} = .90$, .90 is the maximum correlation to be attained.

Where a correlation is determined, the data should include:

(1) The coefficient of correlation.
(2) The probable error of $r$.
(3) The index of reliability.
(4) The standard error of estimate.

(b). *The Interpretation of r.*—It will be noted that the standard error of the measure, $\sigma_M$ and the probable error of the measure, P.E.$_M$, have so far had no part in the formulas presented. It may be stated that[5] "the probable error of measurement bears the same relation to the probable error of estimate that the index of reliability does to the correlation coefficient." The formula for P.E.$_M$ is:

Formula XVI

$$\text{P.E.}_M = .6745\sigma_1\sqrt{1 - r_{1,2}},$$

where $\sigma_1$ and $\sigma_2$ are not equal, then

$$\sigma_1 = \frac{\sigma_1 + \sigma_2}{2}\sqrt{1 - r_{1,2}}.$$

$$\text{The } \sigma_M = \sigma_1\sqrt{1 - r_{1,2}} \ ;$$

and where a test is equally excellent for a wide as for a narrow range,

$$\sigma_1\sqrt{1 - r_{1,2}} = \Sigma\sqrt{1 - R_{1,2}}.$$

$\sigma_1$ = standard deviation for narrow range of traits;
$r_{1.2}$ = reliability coefficient for narrow range of traits;
$\Sigma$ = standard deviation for wide range of traits;
$R_{1.2}$ = reliability coefficient for wide range of traits.

Borrowing from Kelley[6]

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[5]Monroe-Devoss-Kelley, *Educational Tests and Measurement.* Houghton, Mifflin Company, Boston. 1924. Pages 496–497.

[6]Kelley, T. L., *The Reliability of Test Scores.* Journal of Educational Research, May, 1921. Pages 374–379.

Formula XVII $$\frac{\sigma_1}{\Sigma} = \frac{\sqrt{1 - R_{1,2}}}{\sqrt{1 - r_{1,2}}},$$

we may say that a test with an $r = 0.866$ is good. However, it may be that a test is devised for grades V, VI, and VII in arithmetic and the $r$ for the three grades is 0.88. Upon giving the test to the pupils of the fifth grade, the teachers find $r$ is 0.64 and conclude that the achievement of the grade is determined rather poorly. We need to try

$$\Sigma\sqrt{1 - R_{1,2}} = \sigma_1\sqrt{1 - r_{1,2}}.$$

Take at random a problem where $\Sigma$ is 12, $\sigma_1$ is 8, $r$ is .64, then R is .89 and we find that the fifth grade is measured quite satisfactorily.

To interpret properly a reliability coefficient we must have:

(1) Representative sampling;
(2) At least 40 cases;
(3) $r$ and the method of determining it.

## V. Statistical Technique Applied to the New Type Objective Tests

*The Reliability Coefficient.*—The correlation of a set of scores with a good criterion determines the validity of a test. Its reliability is the extent to which the test measures that which it attempts to measure. The reliability coefficient is the coefficient of correlation between two sets of scores secured from:

(1) Two applications of the same test;
(2) Duplicate forms of the test;
(3) Chance halves (odds vs. evens) on a test.

Whenever the coefficient of reliability of a test is determined by two applications of the same test, the formulas as given for the coefficient of correlation will give a satisfactory solution for $r$, the reliability coefficient of the test. However, when $r$ is determined from the chance halves of a test (odds vs. evens), a correction of $r$ must be made.

This correction is effected by means of the Spearman-Brown "Prophecy" Formula, which is:

$$\text{Formula XVIII} \quad r_{n,N} = \frac{nr_{1,2}}{1+(n-1)r_{1,2}},$$

where $n$ is 2; $r_{1,2}$ is the coefficient of correlation between the halves of the test; $r_{n,N}$ is the correlation between scores based on the same material.

Ruch and Stoddard[7] have devised a table for determining $r_{n,N}$ for various values of $r_{1,2}$ and $n$. Whenever the reliability coefficient is determined from chance halves, odds vs. evens, the data should indicate:

(1) The reliability coefficient on chance halves — $r_{1,2}$.
(2) $r_{1,2}$ corrected — $r_{n,N}$.
(3) The probable error of $r$ — P.E.$_r$.
(4) The index of reliability — $r_{1t}$.
(5) The standard error of estimate — $\sigma_{est.}$.

## VI. Determining the Results

It is not always possible to measure or score all pupils in one particular curve. The variation from the true measure or its unreliability must be determined. In other words, it will be necessary to find the deviation of the mean and the standard deviation. For practical purposes, the mean and the standard deviation of a test fix the meaning of the score units. In experimental procedure there are nearly always two groups, one controlled and the other experimental; or there may be two equated groups.[8] In either case, there will be an identical final test. The results of the test scores may be tabulated, and the conclusion definitely determined according to two approved methods:

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[7]Ruch and Stoddard, *Tests and Measurements in High School Instruction*. World Book Company, Yonkers-on-Hudson. 1927. Page 360.

[8]McCall, W. A., *How to Experiment in Education*. Macmillan Co., New York. 1923. Pages 52, 140, 141.

1. *The Experimental Coefficient.*—McCall[9] makes use of the unreliability of the differences. It is true that the differences between the measures determine the conclusion from an experiment, and the unreliability of the difference determines the value of the conclusion. In the experimental coefficient formula, $E.C. = D/2.78\sigma_{diff.}$, D is equal to $M_1 - M_2$ and the standard deviation difference,

$$\sigma_{diff.} = \sqrt{\sigma^2_{av._1} + \sigma^2_{av._2}}$$

or to be more easily recognized,

$$\sigma_{diff.} = \sqrt{\left(\frac{\sigma_1}{\sqrt{N}}\right)^2 + \left(\frac{\sigma_2}{\sqrt{N}}\right)^2}.$$

Hence the formula for the experimental coefficient is

Formula XIX

$$E.C. = \frac{M_1 - M_2}{2.78\sqrt{\left(\frac{\sigma_1}{\sqrt{N}}\right)^2 + \left(\frac{\sigma_2}{\sqrt{N}}\right)^2}}$$

Since all reliability measures give terms in chance, E.C. is always expressed in chance; and tables[10] to convert an experimental coefficient into a statement of chance have been devised.

2. *The Z-score or the Standard Measure Method.*[10]—"The same proportion as between the standard deviation holds for all equivalent deviations from the mean." Where

$$z_1 = \frac{T_1 - M_1}{1} \text{ and } z_2 = \frac{T_2 - M_2}{2},$$

[9]McCall, W. A., *How to Measure in Education.* Macmillan Co., New York. 1922. Page 405.

[10]Kelley, T. L., *Statistical Method.* Macmillan Co., New York. 1924. Pages 114–118.

$z_1$ may be compared with $z_2$. The formula for the $z$-scores is expressed thus:

Formula XX $$\text{Z-score} = \frac{\text{T} - \text{M}}{\sigma},$$

where T is a pupil's score; M is the mean of the distribution; and $\sigma$ is the standard deviation of the distribution. Every score for each distribution is to be converted into a $z$-score, and then the means of the newly established $z$-scores for the groups can be compared definitely, and the results stated quite positively.

These formulas comprise the mathematical background for educational research and should be given such study, discussion, and application as will lead to a higher degree of uniformity in practice and procedure.

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## ASSOCIATION BETWEEN "PASSING" ON FRESHMAN HIGH-SCHOOL TESTS AND THE FIRST SEMESTER IN THE UNIVERSITY OF TEXAS

Last October the freshman classes in mathematics in The University of Texas were given tests on elementary high-school algebra and geometry. The results of these tests were compiled and a general report on them was given in the last number of the *Mathematics Teachers' Bulletin.*

The question has been raised many times as to whether there is any connection between failures on these examinations on high-school mathematics and failures on the freshman courses in mathematics in The University of Texas. Hence a few comparisons based for the most part on G. U. Yule's *Introduction to the Theory of Statistics* may be of general interest.

| | B | $\beta$ |
|---|---|---|
| | 419 | 287 |
| A | (A B) | (A $\beta$) |
| 246 | 204 | 42 |
| $\alpha$ | ($\alpha$ B) | ($\alpha$ $\beta$) |
| 460 | 215 | 245 |

The method of compiling the accompanying table is obvious. Information was available for 706 students. "A" means "Passed on Entrance Examination"; "B" means "Passed on Fall (First) Semester for 1929–30"; while "$\alpha$" and "$\beta$" indicate corresponding "failures to pass." Of the 706 under consideration, there were 246 who passed on the high-school examination, and 460 who failed on it; while 419 of the 706 passed for the semester, 287 failing and the number of (A B), that is the number who passed on the high-school examination and also for the semester is 204. There are 42 who passed the high-school test, and failed for the semester. The ($\alpha$ $\beta$), that is, those who failed on the high-school test and passed for the semester, number 215. Those failing in both cases are 245.

A few simple percentages may not be amiss. In the first place, the ratio

$$\frac{(A\,B)}{(A)} = \frac{204}{246} = .829$$

seems to indicate that a person passing on the high-school examination is quite likely to pass on the semester. On the other hand

$$\frac{(a\,B)}{(a)} = \frac{215}{460} = .467$$

seems to imply that a person who fails the former need not be entirely discouraged as to his semester grade. Again, the per cent of the 706 who passed the high-school test,

$$\frac{(A)}{N} = \frac{246}{706}$$

is practically 35, while the per cent who pass for the semester

$$\frac{(B)}{N} = \frac{419}{706}$$

is 59, a much larger number, fortunately for the student.

According to Yule's formula if the two attributes under discussion, passing the high-school test and for the semester, are independent,

$$\frac{(A\,B)}{(B)} = \frac{(A)}{N},$$

or $(AB)N = AB$. But $204 \times 706 \neq 246 \times 419$. Another way of making the same statement is

$$\frac{AB}{N} = \frac{246 \times 419}{706} = 146 = (AB)$$

if the two "passes" are unrelated, but the (A B) of the table is 204, hence dependence is indicated.

Two other preliminary tests employed by Yule to discover positive association are as follows:

(1) $$\text{Is } \frac{(A\,B)}{(A)} > \frac{(\alpha\,B)}{(\alpha)}?$$

(2) $$\text{Is } \frac{(A\,B)}{(B)} > \frac{(A\beta)}{(\beta)}?$$

Now constructing the ratios in (1), we have

$$\frac{(A\,B)}{(A)} = \frac{204}{246} = .8296$$

$$\frac{(\alpha\,B)}{(\alpha)} = \frac{215}{460} = .4674.$$

Since $.8296 > .4674$, positive association exists between passing on the high-school test and for the semester. Again, using (2) above,

$$\frac{(A\,B)}{(B)} = \frac{204}{419} = .4964,$$

$$\frac{(A\,\beta)}{(\beta)} = \frac{42}{287} = .1463,$$

and $\dfrac{(A\,B)}{B} > \dfrac{(A\,\beta)}{\beta}$, indicating positive association.

Yule's formula for the coefficient of association, which he calls Q, can be written thus:

$$Q = \frac{(A\,B)\,(\alpha\,\beta) - (A\,\beta)\,(\alpha\,B)}{(A\,B)\,(\alpha\,\beta) + (A\,\beta)\,(\alpha\,B)}$$

From the table above,

$$Q = \frac{204 \times 245 - 42 \times 215}{204 \times 245 + 42 \times 215} = .694,$$

a fairly large coefficient of association.

Before concluding this discussion, it may be well to make a few general remarks concerning certain numbers in the table. The values for $(A\,B)$ and $(\alpha\,\beta)$ are not surprising,

but the fact that ($\alpha$B) is about the same size is at first unexpected. It seems rather strange that 215 students failing on the high-school test should pass the semester unless several other considerations are taken into account. Someone might suggest in explanation that the high-school test was difficult. However, many high school and junior high school teachers insist that such is not the case. Undoubtedly some of the freshmen fail on the high-test test because they are ill at ease in the new surroundings, and later during the 18 weeks of the first semester, they become adjusted to their situation. Moreover, numbers of these students have not studied algebra for several years and hence cannot at first recall the processes with which they were once familiar. By reviewing individually and through class work, these students learn the algebra again and can use it in the semester's work. Indeed, the very experience of failing on this high-school test might arouse an earnest student to a realization of his lack of preparation and cause him to overcome it. Again passing or failing implies an arbitrary standard or grade. The passing grade in The University of Texas is 60, and many students made grades between 50 and 60. Obviously these students are only slightly inferior, perhaps not at all really inferior, to students who made a grade of 60 to 65. The longer term of 18 weeks tended to reveal the real standing of these $\alpha$ students. This latter statement will also account for some of the 42 members of the (A$\beta$) class.